International Journal of Mathematics, Statistics and Operations Research
Vol. 3; Number 2; 2023, pp. 221-234
https://doi.org/10.47509/IJMSOR.2023.v03i02.03

ARF INDIA
Academic Open Access Publishing
www.arfjournals.com

WAVELETS ON A GENERAL BOUNDED DOMAIN

Abdellatif Jouini*[a], Hatem Bibi[b]

[a] *University of Tunis El Manar, Faculty of Sciences of Tunis, 2092 Tunis, Tunisia*
[b] *University of Manouba, Higher School of Digital Economy ESEN, 2010 Tunisia*

Abstract

We construct in this paper multiresolution analysis and the associated wavelet basis on a compact bounded domain of $\mathbb{R}^n$ or a compact Riemannian manifold M of dimension n $(n \in \mathbb{N})$. All bases constructed here are generated by a finite number of basic functions and have location properties. To realize this object, we prove at first some lemmas of algebra and functional analysis. Then, we characterize some functional spaces with new norms.

Keywords: Wavelet, Manifold, Laplacian , Spline.
Mathematics Subject Classification: MSC Code: 42C15; 44A15.

1. Introduction

Wavelet method has a great interest in signal and image processing. The construction of wavelet bases on bounded domains has been an active field for many years and extensively discussed in literature ([1], [5] and [7]). This topic is widely used in many scientific domains as numerical analysis or theoretical physics and successfully applied to many problems in Geomathematics or Geophysics.. The most of constructions are based on the decomposition method, introduced by Z. CIESIELSKI and T. FIEGEL in 1982 ([3] and [4]) to construct spline bases of generalized Sobolev spaces $W_p^k(M)$ $(k \in \mathbb{Z}$ and $1 < p < \infty)$ on a Riemannian manifold M. In 1997, the decomposition method was used by A. COHEN, W. DAHMEN and R. SCHNEIDER ([6]

*Corresponding author.
Email addresses: Abdellatif.jouini@fst.rnu.tn (**Abdellatif Jouini***), hatem.bibi@esen.tn (**Hatem Bibi**)

Article History
Received : 04 August 2022; Revised : 10 September 2023; Accepted : 15 September 2023; Published : 26 December 2023

To cite this paper
Abdellatif Jouini & Hatem Bibi (2023). Wavelets of a General Bounded Domain. *International Journal of Mathematics, Statistics and Operations Research.* 3(2), 221-234.

and [7]) to construct biorthogonal wavelet bases $(\psi_\lambda, \widetilde{\psi}_\lambda)_{\lambda\in\nabla}$ of $\mathbb{L}^2(\Omega)$ where Ω is a bounded domain of $\mathbb{R}^d$ $(d \in \mathbb{N})$; these bases were shown to be bases of Sobolev spaces $H^s(\Omega)$ for $|s| < \frac{3}{2}$. There are others constructions based on the decomposition method as well by A. CANUTO and coworkers [1] and by R. MASSON ([2] and [12]). These bases are continuous but not differentiable and have never been implemented. Moreover, there is a slight difficulty in their presentation, due to notational burden and it is often unclear how to get other regularity Sobolev estimates than for $|s| < \frac{3}{2}$. A. Jouini and P.G Lemarié-Rieusset ([9])constructed in an elementary way two multiresolution analyses on the L-shaped domain which are adapted to higher regularity analysis (namely, to the study of the Sobolev space H^k, $k \in \mathbb{Z}$).

Given M a bounded domain of $\mathbb{R}^n$ or a compact Riemannian manifold of dimension n $(n \in \mathbb{N})$, we would like to construct a multiresolution analysis and the wavelet basis by following the minimization method to construct some special functions. Let Δ a Beltrami's Laplacian operator, x_i a sequence of separated nodes, F_j an increasing sequence and d_j non-negative real numbers satisfying :

i) Any sphere with radius d_j contains at most a point of F_j.

ii) Any sphere with radius $c.d_j$ contains at least a point of F_j.

iii) given a d_j satisfying $\frac{1}{c}d_j \leq d_{j+1} \leq d_j$

Such a sequence exists , we need just to consider diadic points and positive real numbers such that $d_j = \frac{d}{2^j}$ then we have $\frac{1}{c}d_j \leq d_{j+1} \leq d_j$ $(c = \frac{1}{2}$ is convenient), the points (x_j) satisfy $d_{i\neq j}(x_i, x_j) > d$. We construct a collection of functions $\left(\psi_{j+1,\lambda}\right)_{\lambda\in T_j}$ which form an orthonormal basis of W_j satisfying the following location properties

$$\begin{aligned} |\psi_{j+1,\lambda}| &\leq Ce^{-\frac{\gamma d(x,\lambda)}{d_j}} \quad \forall\lambda \in T_j \\ |\frac{\partial^\alpha}{\partial x^\alpha}(\psi_{j+1,\lambda})| &\leq C'e^{-\frac{\gamma d(x,\lambda)}{d_j}} \quad \forall\lambda \in T_j \quad |\alpha| < s - n/2 \end{aligned}$$

and this collection form an orthonormal basis of M with the location properties. Moreover, This wavelet basis is also adapted to the study of the Sobolev space $H^s(M)$ $s \in \mathbb{Z}$). The present construction of orthogonal analyses differs from the previous one in the sense that these analyses are generated by a finite number of simple basic functions and have better stability constants.

The contents of this paper is as follows.

In the second section, we prove equivalent norms in Sobolev spaces on a Riemannian compact manifold. These equivalences have a fundamental impact on our work

In the third section, we study the solution of minimization problem and we construct orthogonal multiscale spaces which are very useful for construction of a multiresolution analysis on the manifold. Next, we prove decay to infinity of splines which are very useful for wavelets.

In the last section, we construct on M a multiresolution analysis and the associated wavelet bases which are generated by a finite number of basic functions.

2. Equivalent Norms in Sobolev spaces

Denote

$$H^s(M) = \{f \in L^2(M), \ \ (-\Delta)^s \in L^2(M)\}.$$

Then $H^s(M)$ is a Hilbert space equipped with the norm $||(I+(-\Delta))^{s/2}f||_2$

Proposition 2.1. . *We have the following equivalence*

$$||f||_{H^s} \sim \Big[c||(-\Delta)^{s/2}f||_2 + c'\Big(\sum_{\lambda}|f(x_\lambda)^2|\Big)^{1/2}\Big]^2 \tag{1}$$

The proof of this result exists in [8].

Proposition 2.2. *Let F be a set of points defined by a partition $F = \bigcup_j F_j$ and*

$$K^s = \{f \in H^s / \forall \lambda \in F, \ f(x_\lambda) = 0\}.$$

If $s > n/2$, the set $(-\triangle)^{s/2}K^s$ is closed in L^2. Moreover, V^s is closed in L^2 and the orthogonal of V^s is $(-\triangle)^{s/2}K^s$.

Proof. First let us show

$$\|\sum_{\lambda} a_\lambda \phi_\lambda\|_{H^s} \ \ \sim \ \ \Big(\sum_{\lambda}|a_\lambda|^2\Big)^{1/2} \tag{2}$$

For an integer s, we have

$$
\begin{aligned}
\|\sum_\lambda a_\lambda \phi_\lambda\|_{H^s} &\sim \sum_{|\alpha|\leq s} \int_V |\triangle^\alpha (\sum_\lambda a_\lambda \phi_\lambda)|^2 \\
&\sim \sum_{|\alpha|\leq s} \sum_\lambda \int_{B(x_\lambda, d/2)} |\triangle^\alpha (\sum_\lambda a_\lambda \phi_\lambda)|^2 \\
&\sim \sum_{|\alpha|\leq s} \sum_\lambda \int_{B(x_\lambda, d/2)} |\triangle^\alpha (a_\lambda \phi_\lambda)|^2 \\
&\sim \sum_\lambda |a_\lambda|^2 \sum_{|\alpha|\leq s} \int |\triangle^\alpha (\phi_\lambda)|^2 \\
&\sim \sum_\lambda |a_\lambda|^2 \|\phi_\lambda\|^2_{H^s} \\
&\sim \sum_\lambda |a_\lambda|^2
\end{aligned}
$$

We also have

$$
\|\sum_\lambda a_\lambda \delta(x - x_\lambda)\|_{H^s} \sim \left(\sum_\lambda |a_\lambda|^2\right)^{1/2} \tag{3}
$$

In fact

$$
\begin{aligned}
\sum_\lambda |a_\lambda|^2 &= \langle \sum_\lambda a_\lambda \delta(x - x_\lambda). \sum_\lambda a_\lambda \phi_\lambda \rangle \\
&\leq \|\sum_\lambda a_\lambda \delta(x - x_\lambda)\|_{H^{-s}} \|\sum_\lambda a_\lambda \delta(x - x_\lambda)\|_{H^s} \\
&\leq \|\sum_\lambda a_\lambda \delta(x - x_\lambda)\|_{H^{-s}} C_s \left(\sum_\lambda |a_\lambda|^2\right)^{1/2} \\
&\leq C_s \|\sum_\lambda a_\lambda \delta(x - x_\lambda)\|_{H^{-s}}
\end{aligned}
$$

Reciprocally, if $s > n/2$ we have by Schwartz inequality

$$\begin{aligned}
|\langle\sum_\lambda a_\lambda\delta(x-x_\lambda), f\rangle| &\leq \left(\sum_\lambda |a_\lambda|^2\right)^{1/2}\left(\sum_\lambda |f(x_\lambda)|^2\right)^{1/2}\\
&\leq \left(\sum_\lambda |a_\lambda|^2\right)^{1/2} C_s\|f\|_{H^s}\\
\|\sum_\lambda a_\lambda\delta(x-x_\lambda)\|_{H^{-s}} &\sim C_s\left(\sum_\lambda |a_\lambda|^2\right)^{1/2}
\end{aligned}$$

This proves the norm equivalence. The closed subspace of H^{-s} of linear combinations of $\delta(x-x_\lambda)$ where the coefficients $(a_\lambda) \in L^2(F)$. Recall

$$V^s = \{f \in l^2,\ (-\triangle)^{s/2}f = \sum_\lambda a\lambda\delta(x-x_\lambda)\}$$

Denote B the subspace of H^{-s} generated by $(\delta(x-x\lambda))$ and the function

$$\begin{array}{rcl} V^s & \longrightarrow & B\\ f & \longmapsto & (-\triangle)^{s/2}f \end{array}$$

is continuous , and V^s is a closed space. Moreover, if $f \in V^s$
$f \in L^2$ then $(-\triangle)^{s/2}f \in H^{-s}$ '

$$\begin{aligned}
(-\triangle)^{s/2}f &\in H^{-s} \quad (f\in L^2)\\
\sum_\lambda |a_\lambda|^2 &\leq \infty \quad (see\ (3))\\
(-\triangle)^{s/2}f &\in H^{-\sigma} \quad (\forall\sigma > n/2)
\end{aligned}$$

because the norm equivalence is established for all $s > n/2$. Then, for $s-\sigma > n/2$, we have

$$(-\triangle f)^{s/2}f \in H^{-(s-\sigma)}$$

. In fact, we have

$$\begin{aligned}
\|f\|_{H^\sigma} &\sim \left(\int_0^\infty dE_\lambda(f,f)\right)^{1/2} + \left(\int_0^\infty \lambda^\sigma dE_\lambda(f,f)\right)^{1/2} \\
\|(\triangle)^{s/2} f\|_{H^{-(s-\sigma)}} &\sim \left(\int_0^\infty \lambda^s dE_\lambda(f,f)\right)^{1/2} + \left(\int_0^\infty \lambda^s \lambda^{-(s-\sigma)} dE_\lambda(f,f)\right)^{1/2} \\
&\sim \left(\int_0^\infty \lambda^s dE_\lambda(f,f)\right)^{1/2} + \left(\int_0^\infty \lambda^\sigma dE_\lambda(f,f)\right)^{1/2} < \\
&< +\infty
\end{aligned}$$

$f \in L^2$, then

$$\|f\|_2 = \left(\int_0^\infty dE_\lambda(f,f)\right)^{1/2} < +\infty$$

and for $(\sigma < s - n/2)$ we have $f \in H^\sigma$ and $f \in H^s$.

Let prove now a stronger result. If $f \in K^s$, then $f(x_\lambda) = 0, \forall\lambda$. From proposition 2.1,we have

$$\|f\|_{H^s} \sim \|(-\triangle)^{s/2} f\|_2$$

and the continuity of the following applications

$$\begin{array}{rcl} H^s & \longrightarrow & \mathbb{C} \\ f & \longmapsto & f(x_\lambda) \end{array}$$

gives that K^s is closed because $K^s = f^{-1}(0)$. Then K^s provided with the norm $\|\|_{H^s}$ is a Banach space. The following application

$$\begin{array}{rcl} K^s & \longrightarrow & (\triangle)^{s/2} K^s \\ f & \longmapsto & (-\triangle)^{s/2} f \end{array}$$

is linear, bijective ($f \in L^2$) and bicontinuous because of the norms equivalences (isomorphism) then, $(-\triangle)^{s/2} K^s$ provided with the L^2-norm is a Banach space and $(-\triangle)^{s/2} K^s$ is a closed space.
Let us show that V^s and $(-\triangle)^{s/2} K^s$ are orthogonal. If $f \in V^s$ and $g = (-\triangle)^{s/2} h, \quad h \in K^s$, then

$$\begin{aligned}\langle f, g\rangle &= \langle f, (-\triangle)^{s/2}h\rangle \\ &= \langle (-\triangle)^{s/2}f, h\rangle \\ &= \sum_{\lambda} a_{\lambda}|f|\bar{h}(x\lambda) \\ &= 0\end{aligned}$$

because $\bar{h}(x\lambda) = 0$ and $h \in K^s$.
Inversely, if $f \in \left((-\triangle)^{s/2}K^s\right)^{\perp}$ and $g \in D(V)$. For $h \in K^s$, denote

$$h(x) = g(x) - \sum_{\lambda} g(x_{\lambda})\phi(x_{\lambda})$$

then

$$(-\triangle)^{s/2} \in (-\triangle)^{s/2}K^s$$

but

$$f \in \left((-\triangle)^{s/2}K^s\right)^{\perp}$$

then

$$\langle f, (-\triangle)^{s/2}h\rangle = 0$$

and we have

$$\begin{aligned}\langle f, (-\triangle)^{s/2}g\rangle &= \langle f, \sum_{\lambda} g(x_{\lambda})(-\triangle)^{s/2}\phi_{\lambda}(x)\rangle \\ &= \sum_{\lambda}\langle f, (-\triangle)^{s/2}\phi_{\lambda}\rangle\bar{g}(x_{\lambda})\end{aligned}$$

We conclude that

$$(-\triangle)^{s/2}f = \sum_{\lambda}\langle f, (-\triangle)^{s/2}\phi_{\lambda}\rangle\delta(x - x_{\lambda})$$

and $f \in V^s$. Proposition 2.2 is completely proved.

3. The solution of minimization problem

The object of this section is to solve the problem (P : find a function $f \in H^s(M)$ such that $f(x_\lambda) = F(x_\lambda)$ minimizing $\int_M |(-\triangle)^{s/2} f|^2$ for all functions such that $f(x_\lambda) = F(x_\lambda)$ for $F \in H^s(M)$.

Theorem 3.1. *For $s > n/2$, the problem (P) has a unique solution f_0. This solution is the unique element $f_0 \in V^{2s}$ such that*

$$f_0(x_\lambda) = F(x_\lambda)$$

and we have

$$f_0 = \sum_\lambda F(x_\lambda) L_\lambda^s$$

where $L_\lambda^s \in V^{2s}$ is the unique element such that

$$L_\lambda^s(x_j) = \delta_{\lambda,j}.$$

Proof. Suppose that the solution named f_0 of problem P is known, then there exists an other function of H^s such that

$$(f(x_\lambda) = F(x_\lambda)\ \forall\lambda) \iff (f - f_0 \in K^s)$$

From Proposition 2.2, we have

$$L^2 = V^s \overset{\perp}{\oplus} (-\triangle)^{s/2} K^s)$$

then

$$\begin{aligned} (-\triangle)^{s/2} f_0 &= f_{0,1} + f_{0,2} \\ (-\triangle)^{s/2} f &= f_1 + f_2 \end{aligned}$$

but $f - f_0 \in K^s$, then $f_{0,1} - f_1 = 0$. We deduce that for any function satisfying $f(x_\lambda) = F(x_\lambda)$, its projection in V^s is fixed and

$$\forall \lambda \in \mathbb{C}, \forall h \in K^s$$

$$\|(-\triangle)^{s/2} f_0 + \lambda h\|_2^2 = \|(-\triangle)^{s/2} f_0\|_2^2 + 2Re\Bigg(\lambda \langle (-\triangle)^{s/2} f_0, (-\triangle)^{s/2} h\rangle\Bigg) + |\lambda|^2$$

Then, $\|(-\triangle)^{s/2} f_0\|_2$ is minimal if and only if $\|f_{0,2}\|_2 = 0$ or equivalently

$$(-\triangle)^{s/2} f_0 \in \left((-\triangle)^{s/2} K^s \right)^{\perp}$$

then

$$(-\triangle)^{s/2} f_0 \in V^s \Rightarrow f_0 \in V^{2s}$$

We have

$$\begin{aligned} (-\triangle)^{s/2} f_0 &\in \left((-\triangle)^{s/2} K^s \right)^{\perp} \\ (-\triangle)^{s/2}(F - f_0) &\in (-\triangle)^{s/2} K^s \end{aligned}$$

then, $(-\triangle)^{s/2}(F-f_0)$ is the orthogonal projection of $(-\triangle)^{s/2}K$ on $(-\triangle)^{s/2}K^s$. This result provides a way to obtain f_0 as follows : We derive F in $(-\triangle)^{s/2}F$, then we project it on $(-\triangle)^{s/2}K^s$ on a function $(-\triangle)^{s/2}h$, we get $h = F - f_0$ which implies $f_0 = F - h$.

The uniqueness of f_0 comes from the injectivity of the application

$$(-\triangle)^{s/2} : H^s \longrightarrow L^2$$

and $(-\triangle)^{s/2}$ is an isometry between $B^{s/2}$ and L^2 where

$$B^{s/2} = \{f \in \xi_0^{'}, | \, (-\triangle)^{s/2} f \in L^2$$

the Beppo Levi space. We proved existence and uniqueness of f_0. For $f_0 \in V^{2s}$, let us study the structure of this space. We proved that

$$V^s \subset H^\sigma, \forall \sigma < s - \frac{n}{2}$$

then

$$V^{2s} \subset H^s$$

due to $s < 2s - n/2$. We have $V^{2s} \subset L^2 \subset H^s$, then and as a consequence, the norms $\| \ \|_2$ and $\| \ \|_{H^s}$ are equivalent on V^{2s} to $\left(\sum_\lambda |f(x_\lambda)|^2 \right)^{1/2}$. In fact, we have due to equivalence 1

$$\left(\sum_\lambda |f(x_\lambda)|^2 \right)^{1/2} \leq C_s \|f\|_{H^s}$$

and due also to equivalence 1

$$\|f\|_2 \leq C\left[\left(\sum_\lambda |f(x_\lambda)|^2\right)^{1/2} + \|(-\triangle)^{s/2} f\|_2\right]$$

We have

$$\left(\phi_\lambda(x_j) = \delta_{\lambda,j}\right) \Rightarrow \left((f(x) - \sum_\lambda f(x_\lambda)\phi_\lambda(x)) \in K^s\right)$$

and

$$\begin{aligned} (-\triangle)^{s/2} f - (-\triangle^{s/2}(\sum_\lambda f(x_\lambda)\phi(x_\lambda)) \quad &\in \quad K^s \\ (-\triangle)^{s/2} f \quad &\in \quad V^s \ \text{ if } f \in V^{2s} \end{aligned}$$

then, $(-\triangle)^{s/2} f$ is the orthogonal projection of $(-\triangle^{s/2}(\sum_\lambda f(x_\lambda)\phi(x_\lambda))^\perp$ on $\left((-\triangle)^{s/2} K^s\right)^\perp$ Then, we have control of $\|(-\triangle)^{s/2} f\|_2$ by

$$\|(-\triangle^{s/2}(\sum_\lambda f(x_\lambda)\phi(x_\lambda))\|_2 \leq |(\sum_\lambda |f(x_\lambda)|)|$$

Consequently, we have on M equivalences

$$\|f\|_2 \sim \|f\|_{H^s} \sim \left(\sum_\lambda |f(x_\lambda)|^2\right)^{1/2}$$

We deduce that on V^{2s} we have $f(x) = \sum_\lambda f(x_\lambda) L_\lambda(x)$ where $L^s_\lambda(x_j) = \delta_{\lambda,j}$. ■ The functions L^s_λ satisfy the following estimates.

Theorem 3.2. *let $s \in \mathbb{N}$ then the function L^s_λ and its m-order derivatives are fast decreasing at infinity ($m < s - n/2$) and for $|\alpha| < s - n/2$, we have*

$$|(\frac{\partial}{\partial x})^\alpha L^s_\lambda| \leq C e^{-pd(x_\lambda, x)}$$

where (c, p) are two positive constants independent of α.

This Theorem is proved in [8].

4. Multiresolution analysis and associated wavelets

In this section , we construct at first a multiresolution analysis on the manifold. Next, we describe the associated wavelet basis. Let s be an even integer such as $s > \frac{n}{2}$. Define

$$V_j = \{f \in L^2/(-\Delta)^s f = \sum_{\lambda \in F_j} C_\lambda \delta_\lambda\}$$

. We have the following properties:

i) V_j is a closed subspace of L^2 due Proposition 2.2

ii) $V_j \subset V_{j+1}$ due to $F_j \subset F_{j+1}$

iii) $\cap V_j = \{0\}\}$.

In fact , if $f \in \cap_j V_j$ then $(-\Delta)^s f$ should be a sum of dirac masses at points $\cap_j F_j = \{g\}$ (any translated point) then $(-\Delta)^s f = c\delta_0$ however $\delta_0 \notin (-\Delta)^s L^2$ then $f = 0$. We also have that $\cup_j V_j$ is dense in L^2, in-fact if $f \in (\cup_j V_j)^\perp = \cap_j V_j^\perp$ then $f = (-\Delta)^s h$ where $h \in H^s$ must be null at the points $\cup_j F_j$ (because $V_j^\perp = (-\Delta)^s K^s_{F_j}$) which is dense in V. As a consequence, $h = 0$ and $f = 0$

We want to construct an orthonormal basis of W_j. If $f \in V_j$ then $f = \sum_{\lambda \in F_j} a(\lambda) L^s_{j,\lambda}$ where $L^s_{j,\lambda}$ is the unique element of V_j verifying

$$L^s_{j,\lambda}(\lambda') = \delta_{\lambda,\lambda'}, \forall \lambda' \in F_j$$

where $L^s_{j,\lambda}$ are the Lagrangian interpolation splines functions, then $\left(L^s_{j,\lambda}\right)_{\lambda \in F_j}$ is a basis of V_j. For V_{j+1}, we add the points $\lambda \in F_{j+1}, \lambda \notin F_j$ We begin orthonormalization of $(L^s_{j,\lambda}) \in F_j$ by Gram-Schmidt, we obtain an orthonormal basis $(\phi^s_{j,\lambda})$ of V_j. Theses functions have the same properties of location as $(L^s_{j,\lambda})$ by the matrix computation lemma. Let consider a natural supplementary space $\tilde{W}_j$ of V_j, then we have

$$f \in \tilde{W}_j \text{ if } \; f(\lambda) = 0, \forall \lambda \in F_j$$

This implies that all $f \in V_{j+1}$ has a unique decomposition $f = g + h$ where

$$g \in V_j$$

and $f(\lambda) = g(\lambda),\ \forall\lambda \in F_j$ and $h \in \tilde{W}_j. We have also to$
$\|g\|_2 \leq c\Big(\sum_{\lambda\in F_j} |g(\lambda)|^2\Big)^{1/2}$
$\leq c\Big(\sum_{\lambda\in F_{j+1}} |f(\lambda)|^2\Big)^{1/2}$
$\leq c'\|f\|_2$Then, we get $V_{j+1} = V_j \oplus \tilde{W}_j$. This sum is direct and non orthogonal. We consider the following basis of $\tilde{W}_j$ given by $\Big(L^s_{j+1,\lambda}\Big)_{\lambda\in F_{j+1}|F_j}$.

Denote

$$T_j = F_{j+1}|F_j$$

We project orthogonally on W_j the functions $\Big(L_{j+1,\lambda}\Big)_{\lambda\in T_j}$, then we have

$$\Lambda'_{J+1,\lambda} = L^s_{j+1,\lambda} - \sum_{\lambda'\in F_j} \Big(L^s_{j+1,\lambda'}\phi'_{j',\lambda'}\Big)\phi_{j',\lambda'} \tag{4}$$

The kernel of the orthogonal projection of $P_j \in L^2(M)$ on V_j is given by

$$k(x,y) = \sum_{\lambda\in F_j} \phi^\lambda_j(x) \otimes \phi^\lambda_j(y)$$

If $f \in V_{j+1}$, then

$$\begin{aligned} f &= P_i f + Q_i f \\ \Longrightarrow Id &= P_i + Q_i \end{aligned}$$

Where Q_i is the orthogonal projection on W_j. Then, $Q_j = Id - P_j$. recall (by theorem 3.2) the properties

$$|\phi^\lambda_j| \leq Ce^{-\frac{\gamma d(x,\lambda)}{d_j}}\ and\ |L^\lambda_{j+1}| \leq Ce^{-\frac{\gamma d(x,\lambda)}{d_{j+1}}}$$

These properties are also valid for the derivatives of order less than $s - n/2$. We have

$$\Lambda^s_{j+1,\lambda} = L^s_{j+1,\lambda} - \sum_{\lambda'\in F_j} \langle L^s_{j+1,\lambda}, \phi^s_{j,\lambda'}\rangle \phi^s_{j,\lambda'}$$

Then, $\Lambda^s_{j+1,\lambda}$ and $L^s_{j+1,\lambda}$ have the same estimations. The functions $\Big(\Lambda^s_{j+1,\lambda}\Big)_{\lambda\in T_j = F_{j+1}|F_j}$ form a basis of W_i. We just need to orthonormalize the functions $\Big(\Lambda^s_{j+1,\lambda}\Big)_{\lambda\in T_j = F_{j+1}|F_j}$. Denote the matrix

$$G = \langle \Lambda^s_{j+1,\lambda}, \Lambda^s_{j+1,\lambda'}\rangle_{\lambda,\lambda'\in T_j} = \alpha_{\lambda,\lambda'\in T_j}$$

and

$$\sum \alpha_{\lambda,\lambda'} f_\lambda \bar{f}_{\lambda'} = \sum_{\lambda,\lambda'} \langle f_\lambda \Lambda_{j+1,\lambda}, f_{\lambda'} \Lambda_{j+1,\lambda'} \rangle = \| \sum \lambda \Lambda^s_{j+1,\lambda} \|_2^2$$

and we have

$$C_1 \big(\sum \lambda^2 \big) \leq \| \sum \lambda \Lambda^s_{j+1,\lambda} \|_2 \leq C_2 \Big(\sum \lambda^2 \Big)$$

G is defined positive. We compute $G^{1/2}$, using the property $G^{-1/2} G^{-12} = G^{-1}$, then the family $\left(\psi^s_{j+1,\lambda} \right)_{\lambda \in T_j}$ where $\psi^s_{j+1,\lambda} = \sum_{\lambda' \in T_j} \mu_{\lambda,\lambda'} \Lambda j + 1, \lambda'$ is an orthonormal basis of W_j. But we have $L^2 = \oplus (W_j)_j, then, we get an orthonormal basis of \mathrm{L}^2(M)$ with property of fast decreasing at infinity. We have the following result.

Theorem 4.1. *We constructed a collection of functions $\left(\psi_{j+1,\lambda} \right)_{\lambda \in T_j}$ which form an orthonormal basis of W_j satisfying the following location properties*

$$|\psi_{j+1,\lambda}| \leq C e^{-\frac{\gamma d(x,\lambda)}{d_j}} \quad \forall \lambda \in T_j$$
$$|\frac{\partial^\alpha}{\partial x^\alpha} (\psi_{j+1,\lambda})| \leq C' e^{-\frac{\gamma d(x,\lambda)}{d_j}} \quad \forall \lambda \in T_j \quad |\alpha| < s - n/2$$

and the assembling forms an orthonormal basis of a Riemannian compact manifold V with the location properties.

References

[1] N. Ajmi, A. Jouini, P.G. Lemarié-Rieusset, *Wavelet bases on a triangle*, Appl. Math. Comp **217** (2011), 9387-9396.

[2] C. Canuto, A. Tabacco, K. Urban, *The wavelet element method*, Preprint 1997.

[3] Z. Ciesielski, T. Figiel, *Spline bases in classical function spaces on compact C^∞ manifolds*, Part I and II, Studia Mathematica, T.LXXVI (1983), pp1-58, pp95-136.

[4] Z. Ciesielski, T. Figiel, *Spline approximation and Besov spaces on compact manifolds*, Studia Mathematica, T.LXXV (1982), pp13-36.

[5] A. Cohen, W. Dahmen, R. De Vore, *Multiscale decomposition on bounded domains*, IGPM, Tech. Rep., 2003.

[6] W. Dahmen, R. Schneider, *Composite wavelet bases for operator equations*, Acta Numerica, 9:55-228, 1997.

[7] W. Dahmen, R. Schneider, *Wavelets on manifolds, Construction and domain decomposition*, SIAM Journal on Mathematic Analysis, vol 31, No 1, pp.184-230, 2000.

[8] A. Jouini, *Constructions de bases d'ondelettes sur les variétés*, Thesis, ORSAY, 1993.

[9] A. Jouini, P.G. Lemarié-Rieusset, *Ondelettes sur un ouvert borné du plan*, Prepublication 46 (1992), ORSAY.

[10] A. Jouini, P.G. Lemarié-Rieusset, *Wavelets on the L-shaped domain*, Physica D254 (2013), pp. 12-17

[11] A. Jouini, M. Kratou, *Wavelets on a manifold,* Journal of Functional Analysis, No 248, (2007), 128-151.

[12] R. Masson, *Méthodes d'ondelettes en simulation numérique pour les problèmes elliptiques et point selle*, Thesis, University of Paris 6, Jan. 1999.

Bibnotes

Abdellatif JOUINI obtained his PhD in Harmonic Analysis in the University of Orsay, Paris, France. He is currently working as a Professor in the Department of Mathematics, Faculty of Sciences of Tunis, Tunisia. He has 30 years of teaching experience. He published several papers in reputed journals and published several books in international publishing houses (Narosa Publishing House, Alpha Sciences Oxford, ition du Net, Collection M...). His areas of interest are Wavelet theory, Harmonic Analysis and Image Processing.

Hatem BIBI obtained his Phd in Hurst parameter estimating by wavelets at El Manar University of Tunis and SAMM laboratory in Sorbonne Paris1 in France. He is an assistant Professor in Higher School of Digital Economy in Tunisia, wavelets and long memory processes are his areas of interest.